Glossary: Demographic parameters

Demography uses parameters to describe three frequent, different ideal-type models of population trends. Since they tend to be rarely used in archaeology on the one hand, and are sometimes expressed differently in the specialist literature on demography on the other, a brief introduction to these three models and their parameters is given here. We essentially follow the introductions by Andrew T. Chamberlain (2006, 19-23), M. Henry H. Stevens (2009), and the fundamental work by Kenneth M. Weiss (1973, 6-9). We use the following notation here:

e (or exp), "Euler's number", basis of natural logarithms, approx. 2.718281828459
ln natural logarithm, logarithmus naturalis
P population size
δP difference between initial population size (P_0) and current population size (P_t or P_1)
t time
δt time span between starting time (t_0) and current time (t_1)
λ geometric growth rate (*linear growth rate*)
r exponential growth rate (*instantaneous growth/exponential growth,* also written as "*rmax*")
k growth rate for logistic growth
C upper growth limit, carrying capacity
The parameters λ, r and k are stated here as the unmodified relative numerical results of the calculation (as is the most common practice) and not expressed as a percentage (as in Kraus 2004, for example), which means the latter values are higher by a factor of 100.

Geometric growth

The geometric growth model describes a population which grows (or declines) linearly, i.e. it increases as a constant proportion of the starting population (P_0) from one time interval to the next. In a graphic representation, geometric growth forms a straight line along the time axis. The key parameter to describe geometric growth is λ (lambda). It is calculated as follows:

$\lambda = (\delta P / P_0) / \delta t$.

When λ is known, the population at time t is calculated as:

$P_t = P_0 * (1 + (\lambda * \delta t))$.

In graphic representations, geometric growth forms a straight line which climbs or falls more or less steeply depending on the intensity of the growth or decline (here e.g. **Fehler! Verweisquelle konnte nicht gefunden werden.**).

Exponential growth

Exponential growth — also called "intrinsic" or "unrestricted" growth — is growth whereby the growth rate "r" is applied to the increased population of the preceding time interval in each case. In other words: the so-called compound interest effect is included in the calculation. As a graphic representation, the growth curve of exponential growth is a line which initially climbs gradually and then more and more steeply along the time axis (e.g. Chamberlain, 2006, Fig. 2.3). r is calculated as:

$r = \ln(P_t) - \ln(P_0) / \delta t$, which can also be expressed as:

$r = \ln(\delta P) / \delta t$.

When r is known, the population at time t is calculated as:

$P_t = P_0 * e^{(r * \delta t)}$.

The mathematical relationship between the parameters λ and r is:
$\ln(\lambda) = r$, or $\lambda = e^r$.

In graphical representations, exponential growth forms a curve which becomes steeper and steeper. When growth is relatively low and for relatively short periods of time — as is the case here for the Merovingian era, for example — this line has only a very slight curvature, which is why the differences between this type of growth and geometric growth are scarcely discernible (e.g. here **Fehler! Verweisquelle konnte nicht gefunden werden.**); but when growth rates are high and especially when considered over long periods of time, the differences are considerable.

Doubling time
A further parameter which is often mentioned in relation to the geometric growth model is the "doubling time", i.e. the time (in years) within which a population has doubled (to $2 * P_0$). The doubling time is calculated as:
$t2 = \ln(2) / r$, or as the equivalent: $t2 = \ln(2) / \ln(\lambda)$
where the value of ln(2) is approx. 0.6931472.

Logistic growth
The model of logistic growth sets out to describe the fact that (exponential) growth does not increase indefinitely, but frequently comes up against a limiting factor, which is described as "C" — the upper growth limit/ carrying capacity, which cannot be exceeded. In a graphic representation, a logistic growth curve is an S-shaped curve with slowly and gradually increasing growth initially (exponential phase), relatively steep almost linear growth in the middle ("linear phase") and which flattens out considerably at the end close to the value of C ("saturation phase") (here e.g. **Fehler! Verweisquelle konnte nicht gefunden werden.**; or Chamberlain, 2006, Fig. 2.3). Logistic growth is described by means of two parameters: k, the growth rate, and C, the upper growth limit; the variables are the time (t) and the initial size of the population (P_o). Unlike geometric and exponential growth, the growth rate (k) is not constant but changes as a function of the upper growth limit. The literature contains several ways for formulating the functional equation:

$P_t = C / (1 + \exp(-k + \delta t)$ - [1]

$\delta P / \delta t = k * P_0 * (C - P_0 / C)$ – [acc. to Verhulst 1838,[2],[3]]

$P_t = k * P_0 * (1 - P_0/C)$ – [acc.to Verhulst 1847,[4]]

$P_t = (P_0 * C) / (P_0 + (C - P_0) * e^{(- C*k*\delta t)})$ – [frequently used notation, e.g. [5]].

The logistic growth rate k can be calculated from the exponential growth rate r:
$k = r * (1 – (P_0 / C))$.

[1] Chamberlain, 2006, 22. Notation with three parameters (C, k, δt), without P_0, since this parameter was implicitly set to zero here.
[2] Encyclopedia Britannica, "Logistic population growth": https://www.britannica.com/science/population-ecology/Logistic-population-growth [28.12.2020].
[3] Bacaër (2008).
[4] Wikipedia, "Logistische Gleichung": https://de.wikipedia.org/wiki/Logistische_Gleichung [28.12.2020].
[5] Wikipedia, „Wachstum (Mathematik)": https://de.wikipedia.org/wiki/Wachstum_(Mathematik) [28.12.2020]. Notation with four parameters, i.e. incl. P_0, so that P_0 — the lower limit — can also take on values above zero.